EXPLICIT CALCULATIONS OF TENSOR PRODUCT COEFFICIENTS FOR E_7

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Abstract

We propose a new method to calculate coupling coefficients of E_7 tensor products. Our method is based on explicit use of E_7 characters in the definition of a tensor product.

When applying Weyl character formula for E_7 Lie algebra, one needs to make sums over 2903040 elements of E_7 Weyl group. To implement such enormous sums, we show we have a way which makes their calculations possible. This will be accomplished by decomposing an E_7 character into 72 participating A_7 characters.

I. INTRODUCTION

Let $G_7 = E_7, A_7$ and Λ, Λ' be two dominant weights of G_7 where $R(\Lambda)$ and $R(\Lambda')$ are corresponding irreducible representations. For general terms, we follow the book of Humphreys [**1**] as ever.

Tensor product of these two irreducible representations is defined by,

$$R(\Lambda) \otimes R(\Lambda') = R(\Lambda + \Lambda') + \sum_{\lambda \in S(\lambda+\lambda')} t(\lambda < \Lambda + \Lambda') \quad R(\lambda) \qquad (I.1)$$

where $S(\lambda + \lambda')$ is the set of $\Lambda + \Lambda'$ subdominants and $t(\lambda < \Lambda + \Lambda')$' s are tensor coupling coefficients. Though Steinberg formula is the best known way [**2**], a natural way to calculate tensor coupling coefficients is also to solve the equation

$$Ch(\Lambda) \otimes Ch(\Lambda') = Ch(\Lambda + \Lambda') + \sum_{\lambda \in S(\lambda+\lambda')} t(\lambda < \Lambda + \Lambda') \quad Ch(\lambda) \qquad (I.2)$$

for tensor coupling coefficients. $Ch(\lambda)$ here is the character of an irreducible representation R(λ) which corresponds to a dominant weight λ and it is defined by the famous Weyl Character formula:

$$Ch(\lambda^+) = \frac{A(\lambda^{++})}{A(\rho_{G_7})} \qquad (I.3)$$

where for a weight μ in general

$$A(\mu) \equiv \sum_{\sigma \in W(G_7)} \epsilon(\sigma) \ e^{\sigma(\mu)} \qquad (I.4)$$

$W(G_7)$ is the Weyl Group of G_7 and each and every element σ is the so-called Weyl reflection while $\epsilon(\sigma)$ denotes its sign and $e^{\sigma(\lambda^{++}})$'s here are known as formal exponentials.Thoroughout this work, we assume λ^{++} denotes a strictly dominant weight defined for a dominant λ^+ by

$$\lambda^{++} \equiv \rho_{G_7} + \lambda^+ \qquad (I.5)$$

where ρ_{G_7} is the Weyl vector of G_7.

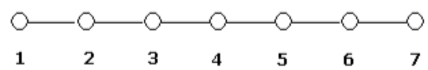
The crucial fact here is that

$$\| W(E_7) \| = 2903040 \qquad (I.6)$$

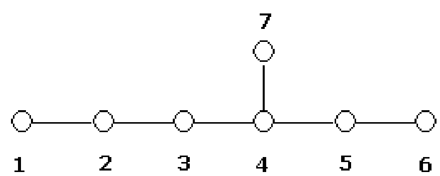
where $\|\mathcal{S}\|$ denotes order of set $\mathcal{S}$. It is easy to see then to implement the sum in (I.4) would not be realizable explicitly. We, instead, propose 72 specifically chosen Weyl reflections which give us A_7 dominant weights participating within the same E_7 Weyl orbit $W(\Lambda^+)$ for any E_7 dominant weight Λ^+. As it is shown in the next section, this makes the evaluation of (I.4) realizable for E_7 but in terms of 72 A_7 characters and hence easily implementable.

II. A_7 DECOMPOSITION OF E_7 LIE ALGEBRA

For $i = 1, 2 \ldots, 7$, let λ_i's and α_i's be respectively the fundamental dominant weigths and simple roots of A_7 Lie algebra with the following Dynkin diagram



where $\rho_{A_7} = \lambda_1 + \ldots + \lambda_7$ is A_7 Weyl vector and Λ_i's be fundamental dominant weights of E_7 Lie algebra in according with the following Dynkin diagram,



where $\rho_{E_7} = \Lambda_1 + \ldots + \Lambda_7$ is E_7 Weyl vector. We suggest following relations allows us to embed A_7 subalgebra into E_7 algebra:

$$\begin{aligned}
\Lambda_1 &= \lambda_2 \\
\Lambda_2 &= \lambda_1 + \lambda_3 \\
\Lambda_3 &= 2\ \lambda_3 \\
\Lambda_4 &= 2\ \lambda_3 + \lambda_6 \\
\Lambda_5 &= \lambda_3 + \lambda_5 \\
\Lambda_6 &= \lambda_4 \\
\Lambda_7 &= \lambda_3 + \lambda_7
\end{aligned} \qquad (II.1)$$

This essentially means that

$$\frac{\|W(E_7)\|}{\|W(A_7)\|} = 72 \qquad (II.2)$$

which tells us that there are at most 72 A_7 dominant weights inside a Weyl orbit $W(\Lambda^+)$. Note here that it is exactly 72 when Λ^+ is a strictly dominant weight. From the now on, $W(\mu)$ will always denotes the Weyl orbit of a weight μ.

As the main point of view of this work, we present in appendix, 72 Weyl reflections to give 72 A_7 dominant weights participating in the same E_7 Weyl orbit $W(\Lambda^+)$ when they are exerted on the dominant weight Λ^+. To this end, the Weyl reflections with respect to simple roots α_i will be called simple reflections σ_i. We extend multiple products of simple reflections trivially by

$$\sigma_{i_1,i_2}(\lambda) \equiv \sigma_{i_1}(\sigma_{i_2}(\lambda)) \ .$$

For $s = 1, \ldots 72$, $\Sigma(s)$'s are 72 Weyl reflections mentioned above. As will also be seen by their definitions that,

$$\begin{aligned} &1) \ \ \epsilon(\sigma(s)) = +1 \quad s = 1, 2, \ldots, 36 \\ &2) \ \ \epsilon(\sigma(s)) = -1 \quad s = 37, 38, \ldots, 72 \end{aligned}$$

III. CALCULATING TENSOR COUPLING COEFFICIENTS

Let us proceed in the instructive example

$$R(\Lambda_3) \otimes R(\Lambda_4) = R(\Lambda_3 + \Lambda_4) + \sum_{j=1}^{39} m(j) \ R(\theta_j) \qquad (III.1)$$

of (I.1). One can see that there are 39 sub-dominant weights θ_j of $\Lambda_3 + \Lambda_4$:

$$\begin{aligned}
\theta_1 &= \Lambda_3 + \Lambda_4 \\
\theta_2 &= \Lambda_1 \\
\theta_3 &= \Lambda_7 \\
\theta_4 &= 3\ \Lambda_1 \\
\theta_5 &= \Lambda_1 + 2\ \Lambda_2 \\
\theta_6 &= 2\ \Lambda_1 + \Lambda_3 \\
\theta_7 &= \Lambda_1 + 2 * \Lambda_5 \\
\theta_8 &= \Lambda_1 + \Lambda_4 + \Lambda_6 \\
\theta_9 &= 2\ \Lambda_2 + \Lambda_7 \\
\theta_{10} &= \Lambda_2 + \Lambda_3 + \Lambda_6 \\
\theta_{11} &= \Lambda_2 + \Lambda_5 + \Lambda_7
\end{aligned}$$

$\theta_{12} = \Lambda_3 + 2\ \Lambda_6$

$\theta_{13} = 3\ \Lambda_7$

$\theta_{14} = 3\ \Lambda_1 + \Lambda_6$

$\theta_{15} = \Lambda_1 + \Lambda_2 + 2\ \Lambda_6$

$\theta_{16} = \Lambda_1 + 3\ \Lambda_6$

$\theta_{17} = \Lambda_1 + \Lambda_6 + 2\ \Lambda_7$

$\theta_{18} = \Lambda_1 + \Lambda_2 + \Lambda_5$

$\theta_{19} = \Lambda_1 + \Lambda_3 + \Lambda_7$

$\theta_{20} = 2\ \Lambda_1 + \Lambda_6 + \Lambda_7$

$\theta_{21} = \Lambda_4 + \Lambda_7$

$\theta_{22} = \Lambda_5 + \Lambda_6 + \Lambda_7$

$\theta_{23} = \Lambda_1 + \Lambda_2$

$\theta_{24} = \Lambda_1 + \Lambda_6$

$\theta_{25} = 2\ \Lambda_1 + \Lambda_7$

$\theta_{26} = \Lambda_1 + 2\ \Lambda_7$

$\theta_{27} = \Lambda_2 + \Lambda_3$

$\theta_{28} = \Lambda_3$

$\theta_{29} = \Lambda_3 + \Lambda_5$

$\theta_{30} = 2\ \Lambda_6 + \Lambda_7$

$\theta_{31} = \Lambda_1 + \Lambda_5 + \Lambda_6$

$\theta_{32} = \Lambda_1 + 2\ \Lambda_6$

$\theta_{33} = \Lambda_6 + \Lambda_7$

$\theta_{34} = \Lambda_1 + \Lambda_4$

$\theta_{35} = \Lambda_1 + \Lambda_5$

$\theta_{36} = \Lambda_1 + \Lambda_2 + \Lambda_6$

$\theta_{37} = \Lambda_2 + \Lambda_6 + \Lambda_7$

$\theta_{38} = \Lambda_2 + \Lambda_7$

$\theta_{39} = \Lambda_3 + \Lambda_6$

$\theta_{40} = \Lambda_5 + \Lambda_7$

To this end, we should care about specialization of formal exponentials **[3]**. Let us consider the so-called Fundamental Weights μ_I which are defined for $(I = 1, \ldots, 8)$ as in the following **[4]**:

$$\alpha_i \equiv \mu_i - \mu_{i+1} \quad , \quad (i = 1, \ldots 7.) \tag{III.2}$$

α_i's here are A_7 simple roots mentioned above and the best way to calculate A_7 and hence E_7 characters is to use the specialization in terms of parameters $u_I \equiv e^{\mu_I}$ which are subjects of the condition $\mu_1 + \mu_2 + \ldots + \mu_8 = 0$ or $u_1 u_2 \ldots u_8 = 1$.

To exemplify (I.3) for E_7, we would like to give detailed calculation of $Ch(\Lambda_3+\Lambda_4)$. By applying 72 specifically chosen Weyl reflections on strictly dominant weight $\rho_{E_7} + \Lambda_3 + \Lambda_4$, one can see we have the following decompositions:

$$A(\rho_{E_7} + \Lambda_3 + \Lambda_4) = \sum_{k=1}^{36} Ch(\nu_k) - \sum_{k=37}^{72} Ch(\nu_k) \tag{III.3}$$

where

$$\begin{aligned}
\nu_1 &= \lambda_1 + \lambda_2 + 11\ \lambda_3 + \lambda_4 + \lambda_5 + 2\ \lambda_6 + \lambda_7 \\
\nu_2 &= 2\ \lambda_1 + 2\ \lambda_2 + 8\ \lambda_3 + \lambda_4 + \lambda_5 + 5\ \lambda_6 + \lambda_7 \\
\nu_3 &= 5\ \lambda_1 + \lambda_2 + 7\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 2\ \lambda_6 + 3\ \lambda_7 \\
\nu_4 &= 4\ \lambda_1 + \lambda_2 + 6\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 4\ \lambda_6 + 3\ \lambda_7 \\
\nu_5 &= 2\ \lambda_1 + 4\ \lambda_2 + 4\ \lambda_3 + \lambda_4 + 5\ \lambda_5 + \lambda_6 + 5\ \lambda_7 \\
\nu_6 &= 5\ \lambda_1 + 2\ \lambda_2 + 5\ \lambda_3 + 2\ \lambda_4 + 2\ \lambda_5 + 3\ \lambda_6 + 4\ \lambda_7 \\
\nu_7 &= 5\ \lambda_1 + 2\ \lambda_2 + 5\ \lambda_3 + \lambda_4 + 4\ \lambda_5 + 3\ \lambda_6 + 2\ \lambda_7 \\
\nu_8 &= 7\ \lambda_1 + \lambda_2 + 6\ \lambda_3 + \lambda_4 + 2\ \lambda_5 + 2\ \lambda_6 + 5\ \lambda_7 \\
\nu_9 &= 7\ \lambda_1 + \lambda_2 + 5\ \lambda_3 + 2\ \lambda_4 + 3\ \lambda_5 + 2\ \lambda_6 + 3\ \lambda_7 \\
\nu_{10} &= 2\ \lambda_1 + 2\ \lambda_2 + 4\ \lambda_3 + \lambda_4 + 6\ \lambda_5 + \lambda_6 + 6\ \lambda_7 \\
\nu_{11} &= 3\ \lambda_1 + 3\ \lambda_2 + 3\ \lambda_3 + 2\ \lambda_4 + 4\ \lambda_5 + 2\ \lambda_6 + 6\ \lambda_7 \\
\nu_{12} &= 3\ \lambda_1 + 3\ \lambda_2 + 3\ \lambda_3 + \lambda_4 + 6\ \lambda_5 + 2\ \lambda_6 + 4\ \lambda_7 \\
\nu_{13} &= 6\ \lambda_1 + \lambda_2 + 5\ \lambda_3 + \lambda_4 + 2\ \lambda_5 + 4\ \lambda_6 + 5\ \lambda_7 \\
\nu_{14} &= 6\ \lambda_1 + \lambda_2 + 4\ \lambda_3 + 2\ \lambda_4 + 3\ \lambda_5 + 4\ \lambda_6 + 3\ \lambda_7 \\
\nu_{15} &= \lambda_1 + 6\ \lambda_2 + \lambda_3 + 4\ \lambda_4 + 2\ \lambda_5 + \lambda_6 + 8\ \lambda_7 \\
\nu_{16} &= 4\ \lambda_1 + 4\ \lambda_2 + 3\ \lambda_3 + \lambda_4 + 4\ \lambda_5 + \lambda_6 + 7\ \lambda_7 \\
\nu_{17} &= 4\ \lambda_1 + 4\ \lambda_2 + 2\ \lambda_3 + 2\ \lambda_4 + 5\ \lambda_5 + \lambda_6 + 5\ \lambda_7 \\
\nu_{18} &= \lambda_1 + 6\ \lambda_2 + \lambda_3 + \lambda_4 + 8\ \lambda_5 + \lambda_6 + 2\ \lambda_7
\end{aligned}$$

$$\nu_{19} = 7\ \lambda_1 + 2\ \lambda_2 + 4\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 3\ \lambda_6 + 4\ \lambda_7$$

$$\nu_{20} = 8\ \lambda_1 + 2\ \lambda_2 + 2\ \lambda_3 + 4\ \lambda_4 + \lambda_5 + 5\ \lambda_6 + \lambda_7$$

$$\nu_{21} = 10\ \lambda_1 + \lambda_2 + 3\ \lambda_3 + 3\ \lambda_4 + \lambda_5 + 4\ \lambda_6 + 2\ \lambda_7$$

$$\nu_{22} = 11\ \lambda_1 + \lambda_2 + \lambda_3 + 6\ \lambda_4 + \lambda_5 + 2\ \lambda_6 + \lambda_7$$

$$\nu_{23} = \lambda_1 + 4\ \lambda_2 + \lambda_3 + 4\ \lambda_4 + 3\ \lambda_5 + \lambda_6 + 9\ \lambda_7$$

$$\nu_{24} = 4\ \lambda_1 + 2\ \lambda_2 + 3\ \lambda_3 + \lambda_4 + 5\ \lambda_5 + \lambda_6 + 8\ \lambda_7$$

$$\nu_{25} = 4\ \lambda_1 + 2\ \lambda_2 + 2\ \lambda_3 + 2\ \lambda_4 + 6\ \lambda_5 + \lambda_6 + 6\ \lambda_7$$

$$\nu_{26} = \lambda_1 + 4\ \lambda_2 + \lambda_3 + \lambda_4 + 9\ \lambda_5 + \lambda_6 + 3\ \lambda_7$$

$$\nu_{27} = 2\ \lambda_1 + 5\ \lambda_2 + \lambda_3 + 3\ \lambda_4 + 2\ \lambda_5 + 2\ \lambda_6 + 9\ \lambda_7$$

$$\nu_{28} = 5\ \lambda_1 + 3\ \lambda_2 + 2\ \lambda_3 + \lambda_4 + 5\ \lambda_5 + 2\ \lambda_6 + 6\ \lambda_7$$

$$\nu_{29} = 9\ \lambda_1 + \lambda_2 + 2\ \lambda_3 + 3\ \lambda_4 + \lambda_5 + 6\ \lambda_6 + 2\ \lambda_7$$

$$\nu_{30} = \lambda_1 + 7\ \lambda_2 + \lambda_3 + 2\ \lambda_4 + 2\ \lambda_5 + \lambda_6 + 10\ \lambda_7$$

$$\nu_{31} = 10\ \lambda_1 + 2\ \lambda_2 + 2\ \lambda_3 + 2\ \lambda_4 + \lambda_5 + 6\ \lambda_6 + \lambda_7$$

$$\nu_{32} = 13\ \lambda_1 + \lambda_2 + \lambda_3 + 4\ \lambda_4 + \lambda_5 + 3\ \lambda_6 + \lambda_7$$

$$\nu_{33} = 2\ \lambda_1 + 2\ \lambda_2 + \lambda_3 + 5\ \lambda_4 + \lambda_5 + \lambda_6 + 12\ \lambda_7$$

$$\nu_{34} = \lambda_1 + 5\ \lambda_2 + \lambda_3 + 2\ \lambda_4 + 3\ \lambda_5 + \lambda_6 + 11\ \lambda_7$$

$$\nu_{35} = 16\ \lambda_1 + \lambda_2 + \lambda_3 + 2\ \lambda_4 + 2\ \lambda_5 + \lambda_6 + \lambda_7$$

$$\nu_{36} = \lambda_1 + \lambda_2 + 3\ \lambda_3 + 2\ \lambda_4 + \lambda_5 + \lambda_6 + 15\ \lambda_7$$

$$\nu_{37} = 3\ \lambda_1 + \lambda_2 + 9\ \lambda_3 + \lambda_4 + \lambda_5 + 4\ \lambda_6 + \lambda_7$$

$$\nu_{38} = 2\ \lambda_1 + \lambda_2 + 8\ \lambda_3 + \lambda_4 + \lambda_5 + 6\ \lambda_6 + \lambda_7$$

$$\nu_{39} = 4\ \lambda_1 + 2\ \lambda_2 + 6\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 3\ \lambda_6 + 3\ \lambda_7$$

$$\nu_{40} = 6\ \lambda_1 + \lambda_2 + 6\ \lambda_3 + 2\ \lambda_4 + 2\ \lambda_5 + 2\ \lambda_6 + 4\ \lambda_7$$

$$\nu_{41} = 6\ \lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 4\lambda_5 + 2\lambda_6 + 2\lambda_7$$

$$\nu_{42} = 2\ \lambda_1 + 3\lambda_2 + 4\lambda_3 + \lambda_4 + 5\lambda_5 + 2\lambda_6 + 5\lambda_7$$

$$\nu_{43} = 5\ \lambda_1 + \lambda_2 + 5\lambda_3 + 2\lambda_4 + 2\lambda_5 + 4\lambda_6 + 4\lambda_7$$

$$\nu_{44} = 5\ \lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 4\lambda_5 + 4\lambda_6 + 2\lambda_7$$

$$\nu_{45} = 3\ \lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 + 4\lambda_5 + \lambda_6 + 6\lambda_7$$

$$\nu_{46} = 3\ \lambda_1 + 4\lambda_2 + 3\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 4\lambda_7$$

$$\nu_{47} = 6\ \lambda_1 + 2\lambda_2 + 5\lambda_3 + \lambda_4 + 2\lambda_5 + 3\lambda_6 + 5\lambda_7$$

$$\nu_{48} = 6\ \lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 3\lambda_6 + 3\lambda_7$$

$$\nu_{49} = 8\ \lambda_1 + \lambda_2 + 5\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 2\ \lambda_6 + 4\ \lambda_7$$

$$\nu_{50} = 9\ \lambda_1 + \lambda_2 + 3\ \lambda_3 + 4\ \lambda_4 + \lambda_5 + 4\ \lambda_6 + \lambda_7$$

$$\nu_{51} = 3\ \lambda_1 + 2\ \lambda_2 + 3\ \lambda_3 + 2\ \lambda_4 + 5\ \lambda_5 + \lambda_6 + 7\ \lambda_7$$
$$\nu_{52} = 3\ \lambda_1 + 2\ \lambda_2 + 3\ \lambda_3 + \lambda_4 + 7\ \lambda_5 + \lambda_6 + 5\ \lambda_7$$
$$\nu_{53} = \lambda_1 + 5\ \lambda_2 + \lambda_3 + 4\ \lambda_4 + 2\ \lambda_5 + 2\ \lambda_6 + 8\ \lambda_7$$
$$\nu_{54} = 4\ \lambda_1 + 3\ \lambda_2 + 3\ \lambda_3 + \lambda_4 + 4\ \lambda_5 + 2\ \lambda_6 + 7\ \lambda_7$$
$$\nu_{55} = 4\ \lambda_1 + 3\ \lambda_2 + 2\ \lambda_3 + 2\ \lambda_4 + 5\ \lambda_5 + 2\ \lambda_6 + 5\ \lambda_7$$
$$\nu_{56} = \lambda_1 + 5\ \lambda_2 + \lambda_3 + \lambda_4 + 8\ \lambda_5 + 2\ \lambda_6 + 2\ \lambda_7$$
$$\nu_{57} = 7\ \lambda_1 + \lambda_2 + 4\ \lambda_3 + \lambda_4 + 3\ \lambda_5 + 4\ \lambda_6 + 4\ \lambda_7$$
$$\nu_{58} = 8\ \lambda_1 + \lambda_2 + 2\ \lambda_3 + 4\ \lambda_4 + \lambda_5 + 6\ \lambda_6 + \lambda_7$$
$$\nu_{59} = 2\ \lambda_1 + 6\ \lambda_2 + \lambda_3 + 3\ \lambda_4 + 2\ \lambda_5 + \lambda_6 + 9\ \lambda_7$$
$$\nu_{60} = 5\ \lambda_1 + 4\ \lambda_2 + 2\ \lambda_3 + \lambda_4 + 5\ \lambda_5 + \lambda_6 + 6\ \lambda_7$$
$$\nu_{61} = 9\ \lambda_1 + 2\ \lambda_2 + 2\ \lambda_3 + 3\ \lambda_4 + \lambda_5 + 5\ \lambda_6 + 2\ \lambda_7$$
$$\nu_{62} = 12\ \lambda_1 + \lambda_2 + \lambda_3 + 5\ \lambda_4 + \lambda_5 + 2\ \lambda_6 + 2\ \lambda_7$$
$$\nu_{63} = 11\ \lambda_1 + \lambda_2 + 3\ \lambda_3 + 2\ \lambda_4 + \lambda_5 + 5\ \lambda_6 + \lambda_7$$
$$\nu_{64} = \lambda_1 + 2\lambda_2 + \lambda_3 + 6\ \lambda_4 + \lambda_5 + \lambda_6 + 11\ \lambda_7$$
$$\nu_{65} = 2\ \lambda_1 + 4\ \lambda_2 + \lambda_3 + 3\ \lambda_4 + 3\ \lambda_5 + \lambda_6 + 10\ \lambda_7$$
$$\nu_{66} = 5\ \lambda_1 + 2\ \lambda_2 + 2\ \lambda_3 + \lambda_4 + 6\ \lambda_5 + \lambda_6 + 7\ \lambda_7$$
$$\nu_{67} = \lambda_1 + 2\ \lambda_2 + \lambda_3 + \lambda_4 + 11\ \lambda_5 + \lambda_6 + \lambda_7$$
$$\nu_{68} = \lambda_1 + 6\ \lambda_2 + \lambda_3 + 2\ \lambda_4 + 2\ \lambda_5 + 2\ \lambda_6 + 10\ \lambda_7$$
$$\nu_{69} = 10\ \lambda_1 + \lambda_2 + 2\ \lambda_3 + 2\lambda_4 + \lambda_5 + 7\lambda_6 + \lambda_7$$
$$\nu_{70} = 15\ \lambda_1 + \lambda_2 + \lambda_3 + 2\ \lambda_4 + 3\ \lambda_5 + \lambda_6 + \lambda_7$$
$$\nu_{71} = \lambda_1 + 3\ \lambda_2 + \lambda_3 + 4\ \lambda_4 + \lambda_5 + \lambda_6 + 13\ \lambda_7$$
$$\nu_{72} = \lambda_1 + \lambda_2 + 2\ \lambda_3 + 2\ \lambda_4 + \lambda_5 + \lambda_6 + 16\ \lambda_7$$

A_7 characters $Ch(\nu_k)$'s are defined by

$$A(\rho_{A_7})\ Ch(\nu_k) = \sum_{\sigma \in W(A_7)} \epsilon(\sigma)\ e^{\sigma(\rho_{A_7} + \nu_k)} \qquad (III.4)$$

Note here that $W(A_7)$ is the permutation group of 8 objects.

To display our result here, we use the following specialization of formal exponentials

with only one free parameter x:

$$
\begin{aligned}
u_1 &= 1\\
u_2 &= 2\\
u_3 &= 3\\
u_4 &= 4\\
u_5 &= 5\\
u_6 &= 6\\
u_7 &= x\\
u_8 &= 1/(720\ x)
\end{aligned}
\tag{III.5}
$$

In this specialization, one obtains the following one-parameter characters:

$$
\begin{aligned}
&A(\rho_{A_7}) = -\frac{1}{2^{20}\times 3^{11}\times 5^6\ x^7}\times\\
&(-6+x)\times(-5+x)\times(-4+x)\times(-3+x)\times(-2+x)\times\\
&(-1+x)\times(-1+720\ x)\times(-1+1440\ x)\times(-1+2160\ x)\times\\
&(-1+2880\ x)\times(-1+3600\ x)\times(-1+4320\ x)\times(-1+720\ x^2)
\end{aligned}
$$

$$
\begin{aligned}
&A(\rho_{E_7}) = -\frac{7^3\times 11\times 13^2\times 17^2\times 23\times 29\times 47\times 59^2\times 71\times 89\times 179\times 239\times 359}{2^{40}\times 3^{19}\times 5^9\times x^{10}}\times\\
&(-1+6\ x)\times(-1+8\ x)\times(-1+10\ x)\times(-1+12\ x)^2\times(-1+15\ x)\\
&(-1+18\ x)\times(-1+20\ x)\times(-1+24\ x)^2\times(-1+30\ x)^2\times(-1+36\ x)\\
&(-1+40\ x)\times(-1+48\ x)\times(-1+60\ x)^2\times(-1+72\ x)\times(-1+90\ x)\times(-1+120\ x)
\end{aligned}
$$

$$
\begin{aligned}
&ch(\Lambda_4) = \frac{1}{2^{16}\times 3^8\times 5^4\ x^4}\ (\\
&2^2\times 7\times 29\times 31\times 113\times 25849\ +\\
&3\times 5^3\times 7^2\times 41\times 13469\times 25841\ x+\\
&2^2\times 7^2\times 227\times 1997\times 1004276389\ x^2+\\
&2^2\times 3^4\times 5^2\times 7^4\times 17\times 41\times 13469\times 45307\ x^3+\\
&2^7\times 3^2\times 266944787316406807\ x^4+\\
&2^6\times 3^6\times 5^3\times 7^4\times 17\times 41\times 13469\times 45307\ x^5+\\
&2^{10}\times 3^4\times 5^2\times 7^2\times 227\times 1997\times 1004276389\ x^6+\\
&2^{12}\times 3^7\times 5^6\times 7^2\times 41\times 13469\times 25841\ x^7+\\
&2^{18}\times 3^8\times 5^4\times 7\times 29\times 31\times 113\times 25849\ x^8)
\end{aligned}
$$

$$ch(\Lambda_3+\Lambda_4)=\frac{7}{2^{28}\times 3^{13}\times 5^{6}\ x^{7}}\ ($$
$$2^5\ \ 7^2\times 139\times 40819\times 22523219+$$
$$3\times 296955329011336071883\ \ x\ \ +$$
$$7\times 93629\times 104327\times 20612147800357\ \ x^2\ \ +$$
$$2^2\times 7\times 89\times 509\times 407193532921684756441\ \ x^3\ \ +$$
$$2^2\times 7\times 19\times 1447\times 73091587\times 1489316745532201\ \ x^4\ \ +$$
$$2^4\times 3\times 11^2\times 37\times 1117\times 18045889\times 1661840436868789\ \ x^5\ \ +$$
$$2^4\times 3^2\times 7\times 11^2\times 5237\times 130069\times 351401\times 12440163841487\ \ x^6\ \ +$$
$$2^9\times 3^3\times 11\times 83\times 757\times 4830390973\times 258355213888973\ \ x^7\ \ +$$
$$2^8\times 3^4\times 5\times 7\times 11^2\times 5237\times 130069\times 351401\times 12440163841487\ \ x^8\ \ +$$
$$2^{12}\times 3^5\times 5^2\times 11^2\times 37\times 1117\times 18045889\times 1661840436868789\ \ x^9\ \ +$$
$$2^{14}\times 3^6\times 5^3\times 7\times 19\times 1447\times 73091587\times 1489316745532201\ \ x^{10}\ \ +$$
$$2^{18}\times 3^8\times 5^4\times 7\times 89\times 509\times 407193532921684756441\ \ x^{11}\ \ +$$
$$2^{20}\times 3^{10}\times 5^5\times 7\times 93629\times 104327\times 20612147800357\ \ x^{12}\ \ +$$
$$2^{24}\times 3^{13}\times 5^6\times 296955329011336071883\ \ x^{13}\ \ +$$
$$2^{33}\times 3^{14}\times 5^7\times 7^2\times 139\times 40819\times 22523219\ \ x^{14})$$

$$ch(\theta_2)=\frac{1}{144\ \ x}\ (3\times 7\times 17+5\times 25841\ x+2^4\times 3^3\times 5\times 7\times 17\ x^2)$$

$$ch(\theta_3)=\frac{1}{2^8\times 3^4\times 5\ x^2}\ ($$
$$5\times 25841+3\times 7\times 19\times 83\times 2459\ x+2^2\times 3^3\times 7^2\times 17\times 45307\ x^2+$$
$$2^4\times 3^3\times 5\times 7\times 19\times 83\times 2459\ x^3+2^8\times 3^4\times 5^3\times 25841\ x^4)$$

$$ch(\theta_4)=\frac{1}{2^{12}\times 3^6\times 5^2\ x^3}\ ($$
$$2\times 3^2\times 7\times 251\times 11287\ +$$
$$2\times 7\times 11\times 1634509733\ x\ +$$
$$3^2\times 7^2\times 23\times 73\times 101\times 281\times 5783\ x^2\ +$$
$$41\times 953\times 5987\times 153547507\ x^3\ +$$
$$2^4\times 3^4\times 5\times 7^2\times 23\times 73\times 101\times 281\times 5783\ x^4\ +$$
$$2^9\times 3^4\times 5^2\times 7\times 11\times 1634509733\ x^5\ +$$
$$2^13\times 3^8\times 5^3\times 7\times 251\times 11287\ x^6)$$

$$ch(\theta_5) = \frac{7}{2^{20} \times 3^{10} \times 5^4\ x^5}\ ($$
$$3^4 \times 389 \times 2621 \times 326611 +$$
$$44372305108670731\ x +$$
$$2 \times 3 \times 7 \times 888351682544432651\ x^2 +$$
$$2 \times 68353596809374436688841\ x^3 +$$
$$2^3 \times 3 \times 17 \times 313 \times 2029 \times 4912124122975679\ x^4 +$$
$$2^2 \times 7 \times 17 \times 87942983 \times 1353499222770007\ x^5 +$$
$$2^7 \times 3^3 \times 5 \times 17 \times 313 \times 2029 \times 4912124122975679\ x^6 +$$
$$2^9 \times 3^4 \times 5^2 \times 68353596809374436688841\ x^7 +$$
$$2^{13} \times 3^7 \times 5^3 \times 7 \times 888351682544432651\ x^8 +$$
$$2^{16} \times 3^8 \times 5^4 \times 44372305108670731\ x^9 +$$
$$2^{20} \times 3^{14} \times 5^5 \times 389 \times 2621 \times 326611\ x^{10})$$

$$ch(\theta_6) = \frac{1}{2^{20} \times 3^{10} \times 5^4\ x^5}\ ($$
$$2 \times 3^5 \times 5 \times 7 \times 67 \times 118953227 +$$
$$2 \times 29 \times 191 \times 1259 \times 18041 \times 1205779\ x +$$
$$3 \times 7 \times 419 \times 953 \times 23122934503549\ x^2 +$$
$$7 \times 11 \times 13 \times 3881 \times 9624859 \times 1896757637\ x^3 +$$
$$2^2 \times 3^2 \times 7 \times 111767 \times 14341763 \times 22885698919\ x^4 +$$
$$2^2 \times 3^2 \times 5 \times 31 \times 114376828127 \times 589217525459\ x^5 +$$
$$2^6 \times 3^4 \times 5 \times 7 \times 111767 \times 14341763 \times 22885698919\ x^6 +$$
$$2^8 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 3881 \times 9624859 \times 1896757637\ x^7 +$$
$$2^{12} \times 3^7 \times 5^3 \times 7 \times 419 \times 953 \times 23122934503549\ x^8 +$$
$$2^{17} \times 3^8 \times 5^4 \times 29 \times 191 \times 1259 \times 18041 \times 1205779\ x^9 +$$
$$2^{21} \times 3^{15} \times 5^6 \times 7 \times 67 \times 118953227\ x^{10})$$

$$ch(\theta_7) = \frac{7}{2^{26} \times 3^{14} \times 5^6\ x^7}\ ($$
$2^2 \times 3^3 \times 71 \times 137 \times 897339719 +$

$2^3 \times 3^2 \times 9427181 \times 12037823647\ x +$

$3 \times 5 \times 7^3 \times 6472827206445606859\ x^2 +$

$7 \times 167 \times 1217 \times 3345931489 \times 3986554903\ x^3 +$

$2^3 \times 3^2 \times 1913 \times 2871252117704830244061\ x^4 +$

$2^3 \times 3^2 \times 5 \times 7 \times 31 \times 5504034331596437227418843\ x^5 +$

$2^4 \times 3^5 \times 5^3 \times 157 \times 311 \times 6473 \times 6273427 \times 25424142019\ x^6 +$

$2^{11} \times 3^5 \times 5 \times 11 \times 13 \times 193 \times 38358559 \times 321202718587369\ x^7 +$

$2^8 \times 3^7 \times 5^4 \times 157 \times 311 \times 6473 \times 6273427 \times 25424142019\ x^8 +$

$2^{11} \times 3^6 \times 5^3 \times 7 \times 31 \times 5504034331596437227418843\ x^9 +$

$2^{15} \times 3^8 \times 5^3 \times 1913 \times 2871252117704830244061\ x^{10}+$

$2^{16} \times 3^8 \times 5^4 \times 7 \times 167 \times 1217 \times 3345931489 \times 3986554903\ x^{11}+$

$2^{20} \times 3^{11} \times 5^6 \times 7^3 \times 6472827206445606859\ x^{12}+$

$2^{27} \times 3^{14} \times 5^6 \times 9427181 \times 12037823647\ x^{13}+$

$2^{30} \times 3^{17} \times 5^7 \times 71 \times 137 \times 897339719\ x^{14})$

$$ch(\theta_8) = \frac{7}{2^{26} \times 3^{12} \times 5^6\ x^7}\ ($$
$70782069982080 +$

$2^5 \times 3 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307\ x +$

$3^1 \times 839 \times 2591 \times 566506158976003\ x^2 +$

$5^2 \times 127 \times 743526604818728544311\ x^3 +$

$2^2 \times 5 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893\ x^4 +$

$2^2 \times 1439 \times 6833 \times 57110377 \times 29440768222739\ x^5 +$

$2^8 \times 3^2 \times 79 \times 2833 \times 5435185631 \times 1421253019763\ x^6 +$

$2^6 \times 3^2 \times 5 \times 7 \times 37577587 \times 1652664511 \times 109347145543\ x^7 +$

$2^{12} \times 3^4 \times 5 \times 79 \times 2833 \times 5435185631 \times 1421253019763\ x^8 +$

$2^{10} \times 3^4 \times 5^2 \times 1439 \times 6833 \times 57110377 \times 29440768222739\ x^9 +$

$2^{14} \times 3^6 \times 5^4 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893\ x^{10} +$

$2^{16} \times 3^8 \times 5^6 \times 127 \times 743526604818728544311\ x^{11} +$

$2^{20} \times 3^{11} \times 5^5 \times 839 \times 2591 \times 566506158976003\ x^{12} +$

$2^{29} \times 3^{13} \times 5^6 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307\ x^{13} +$

$2^{35} \times 3^{15} \times 5^8 \times 19 \times 101 \times 2437 \times 7883\ x^{14})$

$$ch(\theta_9) = \frac{1}{2^{23} \times 3^{12} \times 5^5\ x^6}\ ($$

$5 \times 11 \times 1730263 \times 475374719 +$

$3 \times 5 \times 7 \times 11 \times 4933 \times 15316021986509\ x +$

$5 \times 7 \times 13 \times 172831287782587727803 x^2 +$

$3 \times 7^2 \times 205820307753559239132761 x^3 +$

$2^2 \times 3^3 \times 41 \times 707606359 \times 1135814759505821 x^4 +$

$2 \times 3 \times 5 \times 7 \times 41 \times 1252496251 \times 20514266507074489\ x^5 +$

$2^3 \times 3^3 \times 5 \times 67 \times 16903 \times 2819023 \times 2271886853005801\ x^6 +$

$2^5 \times 3^3 \times 5^2 \times 7 \times 41 \times 1252496251 \times 20514266507074489\ x^7 +$

$2^{10} \times 3^7 \times 5^2 \times 41 \times 707606359 \times 1135814759505821\ x^8 +$

$2^{12} \times 3^7 \times 5^3 \times 7^2 \times 205820307753559239132761\ x^9 +$

$2^{16} \times 3^8 \times 5^5 \times 7 \times 13 \times 172831287782587727803\ x^{10} +$

$2^{20} \times 3^{11} \times 5^6 \times 7 \times 11 \times 4933 \times 15316021986509\ x^{11} +$

$2^{24} \times 3^{12} \times 5^7 \times 11 \times 1730263 \times 475374719\ x^{12})$

$$ch(\theta_{10}) = \frac{1}{2^{26} \times 3^{12} \times 5^6\ x^7}\ ($$

$2^2 \times 3 \times 5 \times 7^2 \times 63113 \times 7700299 +$

$3 \times 7^3 \times 45307 \times 63113 \times 7700299\ x +$

$3 \times 5^2 \times 7 \times 17 \times 191 \times 30444055104860819\ x^2 +$

$2^2 \times 13 \times 854020492656685314145399\ x^3 +$

$2^2 \times 3 \times 7 \times 13 \times 137 \times 1619903 \times 40109150434199081\ x^4 +$

$2^4 \times 3 \times 7 \times 13 \times 948542029 \times 239506095510065339\ x^5 +$

$2^4 \times 3 \times 7 \times 11485393 \times 1462787967158516760029\ x^6 +$

$2^6 \times 3^7 \times 5^2 \times 19 \times 61 \times 883 \times 8053 \times 44101 \times 244021 \times 6224627\ x^7 +$

$2^8 \times 3^3 \times 5 \times 7 \times 11485393 \times 1462787967158516760029\ x^8 +$

$2^{12} \times 3^5 \times 5^2 \times 7 \times 13 \times 948542029 \times 239506095510065339\ x^9 +$

$2^{14} \times 3^7 \times 5^3 \times 7 \times 13 \times 137 \times 1619903, 1 \times 40109150434199081\ x^{10} +$

$2^{18} \times 3^8 \times 5^4 \times 13 \times 854020492656685314145399\ x^{11} +$

$2^{20} \times 3^{11} \times 5^7 \times 7 \times 17 \times 191 \times 30444055104860819\ x^{12} +$

$2^{24} \times 3^{13} \times 5^6 \times 7^3 \times 45307 \times 63113 \times 7700299\ x^{13} +$

$2^{30} \times 3^{15} \times 5^8 \times 7^2 \times 63113 \times 7700299\ x^{14})$

$$ch(\theta_{11}) = \frac{11}{2^{26} \times 3^{14} \times 5^6 \times x^7} \; ($$
$3^2 \times 7 \times 233 \times 2439503305003+$

$2515806591561745145557\, x\, +$

$2 \times 3 \times 7 \times 197 \times 479 \times 83919415489238839\, x^2\, +$

$2 \times 3^5 \times 7 \times 257 \times 13163 \times 561199 \times 23985981091\, x^3\, +$

$2^4 \times 3 \times 5^2 \times 7 \times 23 \times 31 \times 4297 \times 5779 \times 1802909 \times 106751927\, x^4\, +$

$2^3 \times 3^3 \times 207797403557 \times 58222225954444073\, x^5\, +$

$2^5 \times 3^3 \times 7 \times 229242634809951352013683077 41\, x^6\, +$

$2^6 \times 3^5 \times 29 \times 10268423472246027440877336011\, x^7\, +$

$2^9 \times 3^5 \times 5 \times 7 \times 22924263480995135201368307741\, x^8\, +$

$2^{11} \times 3^7 \times 5^2 \times 207797403557 \times 58222225954444073\, x^9\, +$

$2^{16} \times 3^7 \times 5^5 \times 7 \times 23 \times 31 \times 4297 \times 5779 \times 1802909 \times 106751927\, x^{10}\, +$

$2^{17} \times 3^{13} \times 5^4 \times 7 \times 257 \times 13163 \times 561199 \times 23985981091\, x^{11}\, +$

$2^{21} \times 3^{11} \times 5^5 \times 7 \times 197 \times 479 \times 83919415489238839\, x^{12}\, +$

$2^{24} \times 3^{12} \times 5^6 \times 2515806591561745145557\, x^{13}\, +$

$2^{28} \times 3^{16} \times 5^7 \times 7 \times 233 \times 2439503305003\, x^{14})$

$$ch(\theta_{12}) = \frac{1}{2^{24} \times 3^{12} \times 5^6 \times x^7} \; ($$
$2^4 \times 3^3 \times 5^2 \times 7^2 \times 41 \times 13469\, +$

$2^2 \times 3^3 \times 5 \times 7^3 \times 41 \times 13469 \times 45307\, x\, +$

$3 \times 7^3 \times 41 \times 13469 \times 115259 \times 666353\, x^2\, +$

$5 \times 2999 \times 75703 \times 67759950443299\, x^3\, +$

$2^2 \times 3 \times 7 \times 41 \times 73^2 \times 367 \times 2736080070054181\, x^4\, +$

$2^2 \times 3^2 \times 7 \times 7685969351914603034407849\, x^5\, +$

$2^4 \times 3^3 \times 5^2 \times 7^2 \times 373 \times 571 \times 973787 \times 1054751382911\, x^6\, +$

$2^6 \times 3^4 \times 5 \times 37 \times 223 \times 311 \times 3767 \times 12319481 \times 1298488019\, x^7\, +$

$2^8 \times 3^5 \times 5^3 \times 7^2 \times 373 \times 571 \times 973787 \times 1054751382911\, x^8\, +$

$2^{10} \times 3^6 \times 5^2 \times 7 \times 7685969351914603034407849\, x^9\, +$

$2^{14} \times 3^7 \times 5^3 \times 7 \times 41 \times 73^2 \times 367 \times 2736080070054181\, x^{10}\, +$

$2^{16} \times 3^8 \times 5^5 \times 2999 \times 75703 \times 67759950443299\, x^{11}\, +$

$2^{20} \times 3^{11} \times 5^5 \times 7^3 \times 41 \times 13469 \times 115259 \times 666353\, x^{12}\, +$

$2^{26} \times 3^{15} \times 5^7 \times 7^3 \times 41 \times 13469 \times 45307\, x^{13}\, +$

$2^{32} \times 3^{17} \times 5^9 \times 7^2 \times 41 \times 13469\, x^{14})$

$$ch(\theta_{13}) = \frac{11}{2^{24} \times 3^{12} \times 5^5 \ x^6} \ ($$
$$3238985337918907+$$
$$3^4 \times 5 \times 7 \times 841610704258279 \ x \ +$$
$$7^2 \times 13 \times 1482820942905285871 \ x^2 \ +$$
$$3^3 \times 7 \times 193 \times 467 \times 5690719 \times 2565345073 \ x^3 \ +$$
$$2^2 \times 3^2 \times 626862641769013044421771 \ x^4 \ +$$
$$2^4 \times 3^3 \times 5 \times 7 \times 7913499443575152887903 \ x^5 \ +$$
$$2^6 \times 3^4 \times 31 \times 149 \times 181 \times 9313661572349696183 \ x^6 \ +$$
$$2^8 \times 3^5 \times 5^2 \times 7 \times 7913499443575152887903 \ x^7 \ +$$
$$2^{10} \times 3^6 \times 5^2 \times 626862641769013044421771 \ x^8 \ +$$
$$2^{12} \times 3^9 \times 5^3 \times 7 \times 193 \times 467 \times 5690719 \times 2565345073 \ x^9 \ +$$
$$2^{16} \times 3^8 \times 5^4 \times 7^2 \times 13 \times 1482820942905285871 \ x^{10} \ +$$
$$2^{20} \times 3^{14} \times 5^6 \times 7 \times 841610704258279 \ x^{11} \ +$$
$$2^{24} \times 3^{12} \times 5^6 \times 3238985337918907 \ x^{12})$$

$$ch(\theta_{14}) = \frac{1}{2^{18} \times 3^8 \times 5^4 \ x^5} \ ($$
$$2^3 \times 3^2 \times 5 \times 7 \times 251 \times 11287 \ +$$
$$2 \times 3^2 \times 7^2 \times 251 \times 11287 \times 45307 \ x \ +$$
$$2 \times 7^2 \times 13 \times 47 \times 5077 \times 280370003 \ x^2 \ +$$
$$1305047 \times 32776442887619 \ x^3 \ +$$
$$7 \times 29 \times 6779285045465352 1231 \ x^4 \ +$$
$$2^2 \times 5 \times 59 \times 557 \times 739 \times 1838570856221483 \ x^5 \ +$$
$$2^4 \times 3^2 \times 5 \times 7 \times 29 \times 67792850454653521231 \ x^6 \ +$$
$$2^8 \times 3^4 \times 5^2 \times 1305047 \times 32776442887619 \ x^7 \ +$$
$$2^{13} \times 3^6 \times 5^3 \times 7^2 \times 13 \times 47 \times 5077 \times 280370003 \ x^8 \ +$$
$$2^{17} \times 3^{10} \times 5^4 \times 7^2 \times 251 \times 11287 \times 45307 \ x^9 \ +$$
$$2^{23} \times 3^{12} \times 5^6 \times 7 \times 251 \times 11287 \ x^{10})$$

$$ch(\theta_{15}) = \frac{7}{2^{24} \times 3^{12} \times 5^{6} \times x^{7}} \; ($$
$2^{10} \times 3^3 \times 5^3 \times 51977 +$

$2^8 \times 3^3 \times 5^2 \times 7 \times 45307 \times 51977\ x + \ +$

$2^6 \times 3 \times 5 \times 7 \times 51977 \times 115259 \times 666353\ x^2 +$

$3^2 \times 861317 \times 4060866137260903\ x^3 +$

$3 \times 5 \times 41 \times 53 \times 16481 \times 24852500017071257\ x^4 +$

$2^3 \times 223 \times 1035354314616125752647209\ x^5 +$

$2^6 \times 3^2 \times 5 \times 495553698751 \times 90995390639929\ x^6 +$

$2^7 \times 3^2 \times 5 \times 7 \times 163 \times 227 \times 99661 \times 331711758757357 91\ x^7 +$

$2^{10} \times 3^4 \times 5^2 \times 495553698751 \times 90995390639929\ x^8 +$

$2^{11} \times 3^4 \times 5^2 \times 223 \times 1035354314616125752647209\ x^9 +$

$2^{12} \times 3^7 \times 5^4 \times 41 \times 53 \times 16481 \times 24852500017071257\ x^{10} +$

$2^{16} \times 3^{10} \times 5^4 \times 861317 \times 4060866137260903\ x^{11} +$

$2^{26} \times 3^{11} \times 5^6 \times 7 \times 51977 \times 115259 \times 666353\ x^{12} +$

$2^{32} \times 3^{15} \times 5^8 \times 7 \times 45307 \times 51977\ x^{13} +$

$2^{38} \times 3^{17} \times 5^{10} \times 51977\ x^{14})$

$$ch(\theta_{16}) = \frac{1}{2^{22} \times 3^{10} \times 5^{6} \times x^{7}} \; ($$
$2^6 \times 3^3 \times 5^3 \times 7 \times 17 +$

$2^4 \times 3^3 \times 5^2 \times 7^2 \times 17 \times 45307\ x +$

$2^2 \times 3 \times 5 \times 7^2 \times 17 \times 115259 \times 666353\ x^2 +$

$3 \times 7^3 \times 17 \times 37 \times 221303 \times 196292779\ x^3 +$

$5 \times 7 \times 29 \times 147145853 \times 90252382399\ x^4 +$

$2^2 \times 5 \times 862147744154290232767 27\ x^5 +$

$2^4 \times 3^4 \times 5^2 \times 7 \times 863 \times 4003 \times 33749 \times 4576169137\ x^6 +$

$2^6 \times 3^3 \times 5^2 \times 23 \times 43 \times 7907 \times 14463096839915333\ x^7 +$

$2^8 \times 3^6 \times 5^3 \times 7 \times 863 \times 4003 \times 33749 \times 4576169137\ x^8 +$

$2^{10} \times 3^4 \times 5^3 \times 862147744154290232767 27\ x^9 +$

$2^{12} \times 3^6 \times 5^4 \times 7 \times 29 \times 147145853 \times 90252382399\ x^{10} +$

$2^{16} \times 3^9 \times 5^4 \times 7^3 \times 17 \times 37 \times 221303 \times 196292779\ x^{11} +$

$2^{22} \times 3^{11} \times 5^6 \times 7^2 \times 17 \times 115259 \times 666353\ x^{12} +$

$2^{28} \times 3^{15} \times 5^8 \times 7^2 \times 17 \times 45307\ x^{13} +$

$2^{34} \times 3^{17} \times 5^{10} \times 7 \times 17\ x^{14})$

$$ch(\theta_{17}) = \frac{7}{2^{26} \times 3^{12} \times 5^6 \times x^7} \ ($$
$2^2 \times 3^2 \times 5 \times 1920917020339 +$

$3^2 \times 7 \times 45307 \times 1920917020339 \ x +$

$2^2 \times 11 \times 1699 \times 20848837 \times 3497751463 \ x^2 +$

$2^3 \times 3 \times 13 \times 31 \times 47 \times 2517821 \times 2229533970737 \ x^3 +$

$5 \times 23 \times 239 \times 2244205335221034651628 3 \ x^4 +$

$2^3 \times 3 \times 11 \times 244186487395311739881589177 \ x^5 +$

$2^8 \times 3^3 \times 11^2 \times 3229 \times 1352409142907613715103 \ x^6 +$

$2^8 \times 3^2 \times 5 \times 11 \times 99368214071063405675670936 7 \ x^7 +$

$2^{12} \times 3^5 \times 5 \times 11^2 \times 3229 \times 1352409142907613715103 \ x^8 +$

$2^{11} \times 3^5 \times 5^2 \times 11 \times 244186487395311739881589177 \ x^9 +$

$2^{12} \times 3^6 \times 5^4 \times 23 \times 239 \times 2244205335221034651628 3 \ x^{10} +$

$2^{19} \times 3^9 \times 5^4 \times 13 \times 31 \times 47 \times 2517821 \times 2229533970737 \ x^{11} +$

$2^{22} \times 3^{10} \times 5^5 \times 11 \times 1699 \times 20848837 \times 3497751463 \ x^{12} +$

$2^{24} \times 3^{14} \times 5^6 \times 7 \times 45307 \times 1920917020339 \ x^{13} +$

$2^{30} \times 3^{16} \times 5^8 \times 1920917020339 \ x^{14})$

$$ch(\theta_{18}) = \frac{1}{2^{22} \times 3^{12} \times 5^5 \times x^6} \ ($$
$2^2 \times 3^3 \times 5 \times 11 \times 36319 \times 4281313 +$

$2^4 \times 3 \times 5 \times 7 \times 11^2 \times 46549 \times 679298969 \ x +$

$3^2 \times 5^2 \times 7 \times 11 \times 41 \times 10712225499148973 \ x^2 +$

$3 \times 5^2 \times 7 \times 569 \times 476209471 \times 24269673079 \ x^3 +$

$2^7 \times 5^2 \times 7 \times 717343331 \times 35775113699503 \ x^4 +$

$2^8 \times 3^2 \times 7 \times 20327 \times 3472171 \times 37024507616807 \ x^5 +$

$2^4 \times 3^2 \times 3301 \times 3253233143634211446037993 \ x^6 +$

$2^{12} \times 3^4 \times 5 \times 7 \times 20327 \times 3472171 \times 37024507616807 \ x^7 +$

$2^{15} \times 3^4 \times 5^4 \times 7 \times 717343331 \times 35775113699503 \ x^8 +$

$2^{12} \times 3^7 \times 5^5 \times 7 \times 569 \times 476209471 \times 24269673079 \ x^9 +$

$2^{16} \times 3^{10} \times 5^6 \times 7 \times 11 \times 41 \times 10712225499148973 \ x^{10} +$

$2^{24} \times 3^{11} \times 5^6 \times 7 \times 11^2 \times 46549 \times 679298969 \ x^{11} +$

$2^{26} \times 3^{15} \times 5^7 \times 11 \times 36319 \times 4281313 \ x^{12})$

$$ch(\theta_{19}) = \frac{7}{2^{24} \times 3^{12} \times 5^5 \ x^6} \ ($$
$$11 \times 19 \times 54132425994601 +$$
$$2^2 \times 3 \times 1873 \times 801641 \times 1539041989 \ x +$$
$$2^3 \times 5 \times 3391 \times 157884472828805717 \ x^2 +$$
$$3 \times 5 \times 41 \times 421 \times 26894683525927701469 \ x^3 +$$
$$2^3 \times 5296547 \times 40807969 \times 561290628359 \ x^4 +$$
$$2^2 \times 3^2 \times 5 \times 7 \times 18542831 \times 2641233933522645623 \ x^5 +$$
$$2^4 \times 3^4 \times 11699291 \times 71109193 \times 2011720082303 \ x^6 +$$
$$2^6 \times 3^4 \times 5^2 \times 7 \times 18542831 \times 2641233933522645623 \ x^7 +$$
$$2^{11} \times 3^4 \times 5^2 \times 5296547 \times 40807969 \times 561290628359 \ x^8 +$$
$$2^{12} \times 3^7 \times 5^4 \times 41 \times 421 \times 26894683525927701469 \ x^9 +$$
$$2^{19} \times 3^8 \times 5^5 \times 3391 \times 157884472828805717 \ x^{10} +$$
$$2^{22} \times 3^{11} \times 5^5 \times 1873 \times 801641 \times 1539041989 \ x^{11} +$$
$$2^{24} \times 3^{12} \times 5^6 \times 11 \times 19 \times 54132425994601 \ x^{12})$$

$$ch(\theta_{20}) = \frac{7}{2^{22} \times 3^{10} \times 5^5 \times x^6} \ ($$
$$2^3 \times 5 \times 11 \times 1634509733 +$$
$$2 \times 7 \times 11 \times 45307 \times 1634509733 \ x +$$
$$3^1 \times 5^2 \times 188673113 \times 883791547 \ x^2 +$$
$$2^5 \times 821 \times 5573 \times 41867209636027 \ x^3 +$$
$$3^4 \times 269 \times 27680957 \times 2625874917649 \ x^4 +$$
$$2^3 \times 3^2 \times 13 \times 29 \times 71 \times 740736039099992156759 \ x^5 +$$
$$2^5 \times 3^4 \times 5 \times 1489 \times 29150634930575939798 9 \ x^6 +$$
$$2^7 \times 3^4 \times 5 \times 13 \times 29 \times 71 \times 740736039099992156759 \ x^7 +$$
$$2^8 \times 3^8 \times 5^2 \times 269 \times 27680957 \times 2625874917649 \ x^8 +$$
$$2^{17} \times 3^6 \times 5^3 \times 821 \times 5573 \times 41867209636027 \ x^9 +$$
$$2^{16} \times 3^9 \times 5^6 \times 188673113 \times 883791547 \ x^{10} +$$
$$2^{21} \times 3^{10} \times 5^5 \times 7 \times 11 \times 45307 \times 1634509733 \ x^{11} +$$
$$2^{27} \times 3^{12} \times 5^7 \times 11 \times 1634509733 \ x^{12})$$

$$ch(\theta_{21}) = \frac{7}{2^{24} \times 3^{12} \times 5^5\ x^6}\ ($$
$$2^3 \times 11 \times 1091 \times 12756480773+$$
$$3 \times 5 \times 7 \times 4954321 \times 8800968989\ x\ +$$
$$3^2 \times 7 \times 106275811 \times 430270161227\ x^2\ +$$
$$2^3 \times 3 \times 5 \times 7^2 \times 13 \times 3137 \times 25147 \times 246251 \times 502781 \times\ \ x^3\ +$$
$$2^2 \times 3^3 \times 82567 \times 255832037 \times 34487228383\ x^4\ +$$
$$2^5 \times 3^3 \times 7 \times 23 \times 28753 \times 45439 \times 24160314328237\ x^5\ +$$
$$2^7 \times 3^5 \times 5 \times 7 \times 23 \times 31 \times 1867 \times 102273597906161063\ x^6\ +$$
$$2^9 \times 3^5 \times 5 \times 7 \times 23 \times 28753 \times 45439 \times 24160314328237\ x^7+$$
$$2^{10} \times 3^7 \times 5^2 \times 82567 \times 255832037 \times 34487228383\ x^8\ +$$
$$2^{15} \times 3^7 \times 5^4 \times 7^2 \times 13 \times 3137 \times 25147 \times 246251 \times 502781\ x^9\ +$$
$$2^{16} \times 3^{10} \times 5^4 \times 7 \times 106275811 \times 430270161227\ x^{10}\ +$$
$$2^{20} \times 3^{11} \times 5^6 \times 7 \times 4954321 \times 8800968989\ x^{11}\ +$$
$$2^{27} \times 3^{12} \times 5^6 \times 11 \times 1091 \times 12756480773\ x^{12})$$

$$ch(\theta_{22}) = \frac{1}{2^{24} \times 3^{12} \times 5^6\ x^7}\ ($$
$$2^2 \times 3^2 \times 5^2 \times 7 \times 23 \times 443 \times 110339\ +$$
$$3^2 \times 5 \times 7^2 \times 23 \times 443 \times 45307 \times 110339\ x\ +$$
$$2 \times 5 \times 7 \times 79 \times 107166946015042819\ x^2\ +$$
$$2 \times 3 \times 7^2 \times 36713 \times 34478412780970831\ x^3\ +$$
$$3^3 \times 7 \times 8069 \times 484086291493596614 53\ x^4\ +$$
$$2^3 \times 3^2 \times 13 \times 17 \times 4099 \times 26525879 \times 4080242157239\ x^5\ +$$
$$2^4 \times 3^3 \times 7 \times 1277014026590062232215108 27\ x^6\ +$$
$$2^6 \times 3^4 \times 5 \times 401 \times 125717793427846612155232 1\ x^7\ +$$
$$2^8 \times 3^5 \times 5 \times 7 \times 1277014026590062232215108 27\ x^8\ +$$
$$2^{11} \times 3^6 \times 5^2 \times 13 \times 17 \times 4099 \times 26525879 \times 4080242157239\ x^9\ +$$
$$2^{12} \times 3^9 \times 5^3 \times 7 \times 8069 \times 484086291493596614 53\ x^{10}+$$
$$2^{17} \times 3^9 \times 5^4 \times 7^2 \times 36713 \times 34478412780970831\ x^{11}\ +$$
$$2^{21} \times 3^{10} \times 5^6 \times 7 \times 79 \times 107166946015042819\ x^{12}\ +$$
$$2^{24} \times 3^{14} \times 5^7 \times 7^2 \times 23 \times 443 \times 45307 \times 110339\ x^{13}\ +$$
$$2^{30} \times 3^{16} \times 5^9 \times 7 \times 23 \times 443 \times 110339\ x^{14})$$

$$ch(\theta_{23}) = \frac{1}{2^{12} \times 3^6 \times 5^2\ x^3}\ ($$
$$2^6 \times 3 \times 5 \times 7 \times 51977\ +$$
$$3^2 \times 37 \times 967 \times 991 \times 1277\ x\ +$$
$$3 \times 5 \times 7^3 \times 37 \times 67 \times 199 \times 58321 x^2\ +$$
$$2^3 \times 11 \times 104090304018661\ x^3\ +$$
$$2^4 \times 3^3 \times 5^2 \times 7^3 \times 37 \times 67 \times 199 \times 58321\ x^4\ +$$
$$2^8 \times 3^6 \times 5^2 \times 37 \times 967 \times 991 \times 1277\ x^5\ +$$
$$2^{18} \times 3^7 \times 5^4 \times 7 \times 51977\ x^6)$$

$$ch(\theta_{24}) = \frac{7}{2^{10} \times 3^4 \times 5^2\ x^3}\ ($$
$$2^2 \times 3 \times 5 \times 17\ +$$
$$3 \times 7 \times 17 \times 45307\ x\ +$$
$$5 \times 7103 \times 192949\ x^2\ +$$
$$2^2 \times 5 \times 29 \times 541 \times 1456541\ x^3\ +$$
$$2^4 \times 3^2 \times 5^2 \times 7103 \times 192949\ x^4\ +$$
$$2^8 \times 3^5 \times 5^2 \times 7 \times 17 \times 45307\ x^5\ +$$
$$2^{14} \times 3^7 \times 5^4 \times 17\ x^6)$$

$$ch(\theta_{25}) = \frac{1}{2^{16} \times 3^8 \times 5^3\ x^4}\ ($$
$$2 \times 7 \times 11 \times 1634509733\ +$$
$$3 \times 5 \times 7 \times 11149 \times 240779587\ x\ +$$
$$2^3 \times 97 \times 5019793 \times 33822727\ x^2\ +$$
$$3^3 \times 7^2 \times 19 \times 1429 \times 862136910847\ x^3\ +$$
$$2^2 \times 3^5 \times 7 \times 11 \times 127 \times 190523 \times 803051407\ x^4\ +$$
$$2^4 \times 3^5 \times 5 \times 7^2 \times 19 \times 1429 \times 862136910847\ x^5\ +$$
$$2^{11} \times 3^4 \times 5^2 \times 97 \times 5019793 \times 33822727\ x^6\ +$$
$$2^{12} \times 3^7 \times 5^4 \times 7 \times 11149 \times 240779587\ x^7\ +$$
$$2^{17} \times 3^8 \times 5^4 \times 7 \times 11 \times 1634509733\ x^8)$$

$$ch(\theta_{26}) = \frac{1}{2^{20} \times 3^{10} \times 5^4\ x^5}\ ($$
$3^2 \times 7 \times 1920917020339 +$

$2^6 \times 1932627887748647\ x +$

$2^2 \times 3 \times 7 \times 687598205349751411\ x^2 +$

$5 \times 7^2 \times 673 \times 77746343167071049\ x^3 +$

$2^2 \times 3^2 \times 7 \times 41 \times 4513 \times 8209 \times 397543 \times 6033427\ x^4 +$

$2^4 \times 3^2 \times 42256259 \times 5662271823471397\ x^5 +$

$2^6 \times 3^4 \times 5 \times 7 \times 41 \times 4513 \times 8209 \times 397543 \times 6033427\ x^6 +$

$2^8 \times 3^4 \times 5^3 \times 7^2 \times 673 \times 77746343167071049\ x^7 +$

$2^{14} \times 3^7 \times 5^3 \times 7 \times 687598205349751411\ x^8 +$

$2^{22} \times 3^8 \times 5^4 \times 1932627887748647\ x^9 +$

$2^{20} \times 3^{12} \times 5^5 \times 7 \times 1920917020339\ x^{10})$

$$ch(\theta_{27}) = \frac{1}{2^{20} \times 3^9 \times 5^4\ x^5}\ ($$
$7^2 \times 63113 \times 7700299 +$

$5^2 \times 7 \times 1699 \times 195399775663\ x +$

$2^3 \times 7 \times 875675732223561979\ x^2 +$

$2^3 \times 3 \times 103 \times 367 \times 5791 \times 125219 \times 13800671\ x^3 +$

$2^4 \times 7 \times 233 \times 95531 \times 6117073 \times 43889063\ x^4 +$

$2^4 \times 3^2 \times 7 \times 31 \times 7905004010787912179 77\ x^5 +$

$2^8 \times 3^2 \times 5 \times 7 \times 233 \times 95531 \times 6117073 \times 43889063\ x^6 +$

$2^{11} \times 3^5 \times 5^2 \times 103 \times 367 \times 5791 \times 125219 \times 13800671\ x^7 +$

$2^{15} \times 3^6 \times 5^3 \times 7 \times 875675732223561979\ x^8 +$

$2^{16} \times 3^8 \times 5^6 \times 7 \times 1699 \times 195399775663\ x^9 +$

$2^{20} \times 3^{10} \times 5^5 \times 7^2 \times 63113 \times 7700299 x^{10})$

$$ch(\theta_{28}) = \frac{7}{2^{12} \times 3^6 \times 5^2\ x^3}\ ($$
$3 \times 7 \times 41 \times 13469 +$

$5 \times 25841 \times 172357\ x +$

$2^2 \times 3 \times 7 \times 17 \times 151 \times 229 \times 76837\ x^2 +$

$2^2 \times 3^2 \times 13 \times 41 \times 8271569177\ x^3 +$

$2^6 \times 3^3 \times 5 \times 7 \times 17 \times 151 \times 229 \times 76837\ x^4 +$

$2^8 \times 3^4 \times 5^3 \times 25841 \times 172357\ x^5 +$

$2^{12} \times 3^7 \times 5^3 \times 7 \times 41 \times 13469\ x^6)$

$$ch(\theta_{29}) = \frac{1}{2^{22} \times 3^{12} \times 5^5\ x^6}\ ($$
$$3^2 \times 7 \times 37 \times 87365639293 +$$
$$3 \times 7 \times 17 \times 25158739 \times 163675601\ x +$$
$$2 \times 7^2 \times 293207 \times 571321 \times 153212833\ x^2 +$$
$$2 \times 3 \times 5^2 \times 7^2 \times 19 \times 4966016564031243349\ x^3 +$$
$$2^2 \times 3^2 \times 79 \times 383 \times 1049 \times 29339 times 2479950110851\ x^4 +$$
$$2^2 \times 3^3 \times 7 \times 259650710821 \times 25089495454453 x^5 +$$
$$2^7 \times 3^4 \times 7^2 \times 3335043084408465974125 09\ x^6 +$$
$$2^6 \times 3^5 \times 5 \times 7 \times 259650710821 \times 25089495454453\ x^7 +$$
$$2^{10} \times 3^6 \times 5^2 \times 79 \times 383 \times 1049 \times 29339 times 2479950110851\ x^8 +$$
$$2^{13} \times 3^7 \times 5^5 \times 7^2 \times 19 \times 4966016564031243349\ x^9 +$$
$$2^{17} \times 3^8 \times 5^4 \times 7^2 \times 293207 \times 571321 \times 153212833\ x^{10} +$$
$$2^{20} \times 3^{11} \times 5^5 \times 7 \times 17 \times 25158739 \times 163675601\ x^{11} +$$
$$2^{24} \times 3^{14} \times 5^6 \times 7 \times 37 \times 87365639293\ x^{12})$$

$$ch(\theta_{30}) = \frac{1}{2^{20} \times 3^{10} \times 5^5\ x^6}\ ($$
$$2^4 \times 3^2 \times 5^3 \times 25841 +$$
$$2^2 \times 3^2 \times 5^2 \times 7 \times 25841 \times 45307\ x +$$
$$5 \times 7 \times 25841 \times 115259 \times 666353\ x^2 +$$
$$3 \times 7 \times 55717 \times 2804293 \times 13315567\ x^3 +$$
$$2^3 \times 3^2 \times 86399 \times 959723737277659\ x^4 +$$
$$2^6 \times 3^3 \times 5 \times 7 \times 19 \times 372825737 \times 955373927\ x^5 +$$
$$2^8 \times 3^4 \times 5 \times 11 \times 59 \times 1627 \times 137055035733221\ x^6 +$$
$$2^{10} \times 3^5 \times 5^2 \times 7 \times 19 \times 372825737 \times 955373927\ x^7 +$$
$$2^{11} \times 3^6 \times 5^2 \times 86399 \times 959723737277659\ x^8 +$$
$$2^{12} \times 3^7 \times 5^3 \times 7 \times 55717 \times 2804293 \times 13315567\ x^9 +$$
$$2^{16} \times 3^8 \times 5^5 \times 7 \times 25841 \times 115259 \times 666353\ x^{10} +$$
$$2^{22} \times 3^{12} \times 5^7 \times 7 \times 25841 \times 45307\ x^{11} +$$
$$2^{28} \times 3^{14} \times 5^9 \times 25841\ x^{12})$$

$$ch(\theta_{31}) = \frac{1}{2^{20} \times 3^{10} \times 5^5\ x^6}\ ($$
$$2^2 \times 3^2 \times 5 \times 71 \times 1558283 +$$
$$3^2 \times 7 \times 71 \times 45307 \times 1558283\ x\ +$$
$$2 \times 3 \times 7^2 \times 1499 \times 3758088575183\ x^2\ +$$
$$2 \times 5^2 \times 7^2 \times 11261 \times 185753 \times 172860353\ x^3\ +$$
$$3^2 \times 5^2 \times 13 \times 29 \times 241069 \times 7378760342027\ x^4\ +$$
$$2^3 \times 3^3 \times 5 \times 7 \times 11 \times 1366667039504408347 69\ x^5\ +$$
$$2^5 \times 3^4 \times 5^2 \times 7^2 \times 1216883197 \times 110293780367\ x^6\ +$$
$$2^7 \times 3^5 \times 5^2 \times 7 \times 11 \times 136666703950440834769\ x^7\ +$$
$$2^8 \times 3^6 \times 5^4 \times 13 \times 29 \times 241069 \times 7378760342027\ x^8\ +$$
$$2^{13} \times 3^6 \times 5^5 \times 7^2 \times 11261 \times 185753 \times 172860353\ x^9\ +$$
$$2^{17} \times 3^9 \times 5^4 \times 7^2 \times 1499 \times 3758088575183\ x^{10}\ +$$
$$2^{20} \times 3^{12} \times 5^5 \times 7 \times 71 \times 45307 \times 1558283\ x^{11}\ +$$
$$2^{26} \times 3^{14} \times 5^7 \times 71 \times 1558283\ x^{12})$$

$$ch(\theta_{32}) = \frac{7}{2^{16} \times 3^8 \times 5^4\ x^5}\ ($$
$$2^4 \times 3^3 \times 5^2 \times 17\ +$$
$$2^2 \times 3^3 \times 5 \times 7 \times 17 \times 45307\ x\ +$$
$$3 \times 7 \times 17 \times 115259 \times 666353\ x^2\ +$$
$$5 \times 2512868354279147\ x^3\ +$$
$$2^2 \times 3^2 \times 5 \times 1061 \times 218249 \times 31482709\ x^4\ +$$
$$2^4 \times 3^2 \times 5^2 \times 7 \times 2069 \times 634759 \times 1884341\ x^5\ +$$
$$2^6 \times 3^4 \times 5^2 \times 1061 \times 218249 \times 31482709\ x^6\ +$$
$$2^8 \times 3^4 \times 5^3 \times 2512868354279147\ x^7\ +$$
$$2^{12} \times 3^7 \times 5^3 \times 7 \times 17 \times 115259 \times 666353\ x^8\ +$$
$$2^{18} \times 3^{11} \times 5^5 \times 7 \times 17 \times 45307\ x^9\ +$$
$$2^{24} \times 3^{13} \times 5^7 \times 17\ x^{10})$$

$$ch(\theta_{33}) = \frac{1}{2^{14} \times 3^6 \times 5^3\ x^4}\ ($$

$2^2 \times 5^2 \times 25841 +$

$5 \times 7 \times 25841 \times 45307\ x +$

$3 \times 7^2 \times 19 \times 83 \times 2459 \times 45307\ x^2 +$

$2^3 \times 3 \times 7^2 \times 43 \times 53841866849\ x^3 +$

$2^5 \times 3^4 \times 5 \times 71 \times 109 \times 401 \times 1543 \times 1789\ x^4 +$

$2^7 \times 3^3 \times 5 \times 7^2 \times 43 \times 53841866849 x^5 +$

$2^8 \times 3^5 \times 5^2 \times 7^2 \times 19 \times 83 \times 2459 \times 45307\ x^6 +$

$2^{12} \times 3^6 \times 5^4 \times 7 \times 25841 \times 45307\ x^7 +$

$2^{18} \times 3^8 \times 5^6 \times 25841\ x^8)$

$$ch(\theta_{34}) = \frac{1}{2^{20} \times 3^{10} \times 5^4\ x^5}\ ($$

$2^5 \times 3 \times 7 \times 19 \times 101 \times 2437 \times 7883+$

$3^3 \times 7 \times 17 \times 43 \times 670034705677\ x +$

$3 \times 5 \times 7^2 \times 1093 \times 72068616146713\ x^2 +$

$2^2 \times 5 \times 7^2 \times 144593 \times 92161130410753\ x^3 +$

$2^2 \times 3^3 \times 7 \times 4127 \times 8389 \times 31543 \times 1323118663\ x^4 +$

$2^7 \times 3^2 \times 11 \times 11943616469 \times 264210902899\ x^5 +$

$2^6 \times 3^5 \times 5 \times 7 \times 4127 \times 8389 \times 31543 \times 1323118663\ x^6 +$

$2^{10} \times 3^4 \times 5^3 \times 7^2 \times 144593 \times 92161130410753\ x^7 +$

$2^{12} \times 3^7 \times 5^4 \times 7^2 \times 1093 \times 72068616146713\ x^8 +$

$2^{16} \times 3^{11} \times 5^4 \times 7 \times 17 \times 43 \times 670034705677\ x^9 +$

$2^{25} \times 3^{11} \times 5^5 \times 7 \times 19 \times 101 \times 2437 \times 7883\ x^{10})$

$$ch(\theta_{35}) = \frac{1}{2^{14} \times 3^8 \times 5^3\ x^4}\ ($$

$3^2 \times 71 \times 1558283 +$

$3 \times 7^2 \times 11 \times 71 \times 57706391\ x +$

$5 \times 7 \times 71 \times 569 \times 15083 \times 158143\ x^2 +$

$3^2 \times 5 \times 7^2 \times 11 \times 76091 \times 256751687\ x^3 +$

$2^6 \times 3^2 \times 5 \times 238247 \times 29004591751\ x^4 +$

$2^4 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 76091 \times 256751687\ x^5 +$

$2^8 \times 3^4 \times 5^3 \times 7 \times 71 \times 569 \times 15083 \times 158143\ x^6 +$

$2^{12} \times 3^7 \times 5^3 \times 7^2 \times 11 \times 71 \times 57706391\ x^7 +$

$2^{16} \times 3^{10} \times 5^4 \times 71 \times 1558283\ x^8)$

$$ch(\theta_{36}) = \frac{1}{2^{18} \times 3^{8} \times 5^{4}\ x^{5}}\ ($$
$$2^{8} \times 3 \times 5^{2} \times 7 \times 51977\ +$$
$$2^{6} \times 3 \times 5 \times 7^{2} \times 45307 \times 51977\ x\ +$$
$$3 \times 7 \times 13 \times 577 \times 696629 \times 1185337\ x^{2}\ +$$
$$5 \times 509 \times 547 \times 37532952421247\ x^{3}\ +$$
$$2^{3} \times 7 \times 255247 \times 401008447405121\ x^{4}\ +$$
$$2^{4} \times 5 \times 53 \times 2861 \times 1461797 \times 14111392523\ x^{5}\ +$$
$$2^{7} \times 3^{2} \times 5 \times 7 \times 255247 \times 401008447405121\ x^{6}\ +$$
$$2^{8} \times 3^{4} \times 5^{3} \times 509 \times 547 \times 37532952421247\ x^{7}+$$
$$2^{12} \times 3^{7} \times 5^{3} \times 7 \times 13 \times 577 \times 696629 \times 1185337\ x^{8}\ +$$
$$2^{22} \times 3^{9} \times 5^{5} \times 7^{2} \times 45307 \times 51977\ x^{9}\ +$$
$$2^{28} \times 3^{11} \times 5^{7} \times 7 \times 51977^{1}\ x^{10})$$

$$ch(\theta_{37}) = \frac{1}{2^{22} \times 3^{10} \times 5^{5}\ x^{6}}\ ($$
$$2^{2} \times 5 \times 7 \times 31 \times 193 \times 3624787\ +$$
$$7^{2} \times 31 \times 193 \times 45307 \times 3624787\ x\ +$$
$$2 \times 31 \times 79 \times 6043 \times 672439 \times 3360061\ x^{2}\ +$$
$$3 \times 5 \times 7^{2} \times 40277 \times 1031399 \times 973746491\ x^{3}\ +$$
$$2^{2} \times 3 \times 5 \times 7 \times 29 \times 43 \times 97 \times 1375243 \times 57552822853\ x^{4}\ +$$
$$2^{4} \times 3 \times 7^{2} \times 239 \times 45007 \times 10354439038651493\ x^{5}\ +$$
$$2^{6} \times 3^{6} \times 5 \times 11 \times 107 \times 163 \times 208977919833364987\ x^{6}\ +$$
$$2^{8} \times 3^{3} \times 5 \times 7^{2} \times 239 \times 45007 \times 10354439038651493\ x^{7}\ +$$
$$2^{10} \times 3^{5} \times 5^{3} \times 7 \times 29 \times 43 \times 97 \times 1375243 \times 57552822853\ x^{8}\ +$$
$$2^{12} \times 3^{7} \times 5^{4} \times 7^{2} \times 40277 \times 1031399 \times 973746491\ x^{9}\ +$$
$$2^{17} \times 3^{8} \times 5^{4} \times 31 \times 79 \times 6043 \times 672439 \times 3360061\ x^{10}\ +$$
$$2^{20} \times 3^{10} \times 5^{5} \times 7^{2} \times 31 \times 193 \times 45307 \times 3624787\ x^{11}\ +$$
$$2^{26} \times 3^{12} \times 5^{7} \times 7 \times 31 \times 193 \times 3624787\ x^{12})$$

$$ch(\theta_{38}) = \frac{7}{2^{16} \times 3^8 \times 5^3 \ x^4} \ ($$
$$31 \times 193 \times 3624787 +$$
$$2 \times 3 \times 661 \times 39827 \times 198031 \ x +$$
$$3^4 \times 5 \times 9371 \times 3530299103 \ x^2 +$$
$$2^2 \times 3 \times 5 \times 91297 \times 232912445897 \ x^3 +$$
$$2^4 \times 3^3 \times 370597 \times 315220711859 \ x^4 +$$
$$2^6 \times 3^3 \times 5^2 \times 91297 \times 232912445897 \ x^5 +$$
$$2^8 \times 3^8 \times 5^3 \times 9371 \times 3530299103 \ x^6 +$$
$$2^{13} \times 3^7 \times 5^3 \times 661 \times 39827 \times 198031 \ x^7 +$$
$$2^{16} \times 3^8 \times 5^4 \times 31 \times 193 \times 3624787 \ x^8)$$

$$ch(\theta_{39}) = \frac{1}{2^{18} \times 3^8 \times 5^4 \times x^5} \ ($$
$$2^2 \times 3 \times 5 \times 7^2 \times 41 \times 13469 +$$
$$3 \times 7^3 \times 41 \times 13469 \times 45307 \ x +$$
$$5 \times 7 \times 19813 \times 26321 \times 2574193 \ x^2 +$$
$$2^2 \times 3 \times 7 \times 41946929 \times 2853739787 \ x^3 +$$
$$2^2 \times 3 \times 7 \times 11 \times 875056131284689657 \ x^4 +$$
$$2^4 \times 3^2 \times 5^2 \times 8352802228682522537 \ x^5 +$$
$$2^6 \times 3^3 \times 5 \times 7 \times 11 \times 875056131284689657 \ x^6 +$$
$$2^{10} \times 3^5 \times 5^2 \times 7 \times 41946929 \times 2853739787 \ x^7 +$$
$$2^{12} \times 3^6 \times 5^4 \times 7 \times 19813 \times 26321 \times 2574193 \ x^8 +$$
$$2^{16} \times 3^9 \times 5^4 \times 7^3 \times 41 \times 13469 \times 45307 \ x^9 +$$
$$2^{22} \times 3^{11} \times 5^6 \times 7^2 \times 41 \times 13469 \ x^{10})$$

$$
\begin{aligned}
ch(\theta_{40}) = & \frac{7}{2^{18} \times 3^{10} \times 5^4 \times x^5} \; (\\
& 3^2 \times 5 \times 23 \times 443 \times 110339 \; + \\
& 5 \times 29 \times 31 \times 113 \times 25841 \times 25849 \; x \; + \\
& 3 \times 7 \times 19 \times 29 \times 31 \times 83 \times 113 \times 2459 \times 25849 \; x^2 \; + \\
& 3^2 \times 7 \times 419 \times 2364953 \times 600815491 \; x^3 \; + \\
& 2^{10} \times 3^3 \times 11 \times 23 \times 8912381 \times 41333989 \; x^4 \; + \\
& 2^4 \times 3^4 \times 61 \times 67 \times 17529680553917933 \; x^5 \; + \\
& 2^{14} \times 3^5 \times 5 \times 11 \times 23 \times 8912381 \times 41333989 \; x^6 \; + \\
& 2^8 \times 3^6 \times 5^2 \times 7 \times 419 \times 2364953 \times 600815491 \; x^7 \; + \\
& 2^{12} \times 3^7 \times 5^3 \times 7 \times 19 \times 29 \times 31 \times 83 \times 113 \times 2459 \times 25849 \; x^8 \; + \\
& 2^{16} \times 3^8 \times 5^5 \times 29 \times 31 \times 113 \times 25841 \times 25849 \; x^9 \; + \\
& 2^{20} \times 3^{12} \times 5^6 \times 23 \times 443 \times 110339 \; x^{10})
\end{aligned}
$$

Now, one can see that the characters above fulfill the following equation:

$$
\begin{aligned}
ch(\Lambda_3) \times ch(\Lambda_4) = & ch(\Lambda_3 + \Lambda_4) \; + \\
& ch(\theta_2) \; + \; ch(\theta_3) \; + \; ch(\theta_4) \; + \; ch(\theta_5) \; + \\
& ch(\theta_6) \; + \; ch(\theta_7) \; + \; ch(\theta_8) \; + \; ch(\theta_9) \; + \\
& ch(\theta_{10}) \; + \; ch(\theta_{11}) \; + \; ch(\theta_{12}) \; + \; ch(\theta_{13}) \; + \\
& ch(\theta_{14}) \; + \; ch(\theta_{15}) \; + \; ch(\theta_{16}) \; + \; ch(\theta_{17}) \; + \\
& 2 \; ch(\theta_{18}) \; + \; 2 \; ch(\theta_{19}) \; + \; 2 \; ch(\theta_{20}) \; + \; 2 \; ch(\theta_{21}) \; + \; 2 \; ch(\theta_{22}) \; + \\
& 3 \; ch(\theta_{23}) \; + \; 3 \; ch(\theta_{24}) \; + \; 3 \; ch(\theta_{25}) \; + 3 \; ch(\theta_{26}) \; + \\
& 3 \; ch(\theta_{27}) \; + \; 3 \; ch(\theta_{28}) \; + \; 3 \; ch(\theta_{29}) \; + \; 3 \; ch(\theta_{30}) \; + \\
& 4 \; ch(\theta_{31}) \; + \; 4 \; ch(\theta_{32}) \; + \; 4 \; ch(\theta_{33}) \; + \\
& 5 \; ch(\theta_{34}) \; + \; 5 \; ch(\theta_{35}) \; + \; 5 \; ch(\theta_{36}) \; + \; 5 \; ch(\theta_{37}) \; + \\
& 5 \; ch(\theta_{38}) \; + \; 5 \; ch(\theta_{39}) \; + \; 5 \; ch(\theta_{40})
\end{aligned}
\qquad (III.6)
$$

One should note however that, the 1-parameter specialization (III.5) above is not enough to find all the tensor coupling coefficients completely so we saw that at least 3-parameters

specializations will be sufficient, which we used the following one;

$$
\begin{aligned}
u_1 &= 1 \\
u_2 &= 2 \\
u_3 &= 3 \\
u_4 &= 4 \\
u_5 &= x \\
u_6 &= y \\
u_7 &= z \\
u_8 &= 1/(24\ x\ y\ z)
\end{aligned}
$$

IV. APPENDIX

$$
\begin{aligned}
&\Sigma(1) = 1 \\
&\Sigma(2) = \sigma_{3,2}\ ,\ \Sigma(3) = \sigma_{3,4} \\
&\Sigma(4) = \sigma_{3,2,1,4}\ ,\ \Sigma(5) = \sigma_{3,2,4,3}\ ,\ \Sigma(6) = \sigma_{3,2,4,5} \\
&\Sigma(7) = \sigma_{3,2,4,7}\ ,\ \Sigma(8) = \sigma_{3,4,5,6}\ ,\ \Sigma(9) = \sigma_{3,4,5,7} \\
&\Sigma(10) = \sigma_{3,2,1,4,3,2}\ ,\ \Sigma(11) = \sigma_{3,2,1,4,3,5}\ ,\ \Sigma(12) = \sigma_{3,2,1,4,3,7} \\
&\Sigma(13) = \sigma_{3,2,1,4,5,6}\ \Sigma(14) = \sigma_{3,2,1,4,5,7}\ ,\ \Sigma(15) = \sigma_{3,2,4,3,5,4} \\
&\Sigma(16) = \sigma_{3,2,4,3,5,6}\ ,\ \Sigma(17) = \sigma_{3,2,4,3,5,7}\ ,\ \Sigma(18) = \sigma_{3,2,4,3,7,4} \\
&\Sigma(19) = \sigma_{3,2,4,5,6,7}\ ,\ \Sigma(20) = \sigma_{3,2,4,5,7,4}\ ,\ \Sigma(21) = \sigma_{3,4,5,6,7,4} \\
&\Sigma(22) = \sigma_{3,4,5,7,4,3} \\
&\Sigma(23) = \sigma_{3,2,1,4,3,2,5,4}\ ,\ \Sigma(24) = \sigma_{3,2,1,4,3,2,5,6}\ ,\ \Sigma(25) = \sigma_{3,2,1,4,3,2,5,7} \\
&\Sigma(26) = \sigma_{3,2,1,4,3,2,7,4}\ ,\ \Sigma(27) = \sigma_{3,2,1,4,3,5,4,6}\ ,\ \Sigma(28) = \sigma_{3,2,1,4,3,5,6,7} \\
&\Sigma(29) = \sigma_{3,2,1,4,5,6,7,4}\ ,\ \Sigma(30) = \sigma_{3,2,4,3,5,4,6,5}\ ,\ \Sigma(31) = \sigma_{3,2,4,5,6,7,4,5} \\
&\Sigma(32) = \sigma_{3,4,5,6,7,4,3,5} \\
&\Sigma(33) = \sigma_{3,2,1,4,3,2,5,4,3,6}\ ,\ \Sigma(34) = \sigma_{3,2,1,4,3,2,5,4,6,5}\ ,\ \Sigma(35) = \sigma_{3,4,5,6,7,4,3,5,4,7} \\
&\Sigma(36) = \sigma_{3,2,1,4,3,2,5,4,3,6,5,4}
\end{aligned}
$$

$$\begin{aligned}
&\Sigma(37) = \sigma_3 \\
&\Sigma(38) = \sigma_{3,2,1} \ , \ \Sigma(39) = \sigma_{3,2,4} \ , \ \Sigma(40) = \sigma_{3,4,5} \ , \ \Sigma(41) = \sigma_{3,4,7} \\
&\Sigma(42) = \sigma_{3,2,1,4,3} \ , \ \Sigma(43) = \sigma_{3,2,1,4,5} \ , \ \Sigma(44) = \sigma_{3,2,1,4,7} \\
&\Sigma(45) = \sigma_{3,2,4,3,5} \ , \ \Sigma(46) = \sigma_{3,2,4,3,7} \ , \ \Sigma(47) = \sigma_{3,2,4,5,6} \\
&\Sigma(48) = \sigma_{3,2,4,5,7} \ , \ \Sigma(49) = \sigma_{3,4,5,6,7} \ , \ \Sigma(50) = \sigma_{3,4,5,7,4} \\
&\Sigma(51) = \sigma_{3,2,1,4,3,2,5} \ , \ \Sigma(52) = \sigma_{3,2,1,4,3,2,7} \ , \ \Sigma(53) = \sigma_{3,2,1,4,3,5,4} \\
&\Sigma(54) = \sigma_{3,2,1,4,3,5,6} \ , \ \Sigma(55) = \sigma_{3,2,1,4,3,5,7} \ , \ \Sigma(56) = \sigma_{3,2,1,4,3,7,4} \ , \\
&\Sigma(57) = \sigma_{3,2,1,4,5,6,7} \ , \ \Sigma(58) = \sigma_{3,2,1,4,5,7,4} \ , \ \Sigma(59) = \sigma_{3,2,4,3,5,4,6} \qquad (II.6) \\
&\Sigma(60) = \sigma_{3,2,4,3,5,6,7} \ , \ \Sigma(61) = \sigma_{3,2,4,5,6,7,4} \ , \ \Sigma(62) = \sigma_{3,4,5,6,7,4,3} \\
&\Sigma(63) = \sigma_{3,4,5,6,7,4,5} \\
&\Sigma(64) = \sigma_{3,2,1,4,3,2,5,4,3} \ , \ \Sigma(65) = \sigma_{3,2,1,4,3,2,5,4,6} \ , \ \Sigma(66) = \sigma_{3,2,1,4,3,2,5,6,7} \\
&\Sigma(67) = \sigma_{3,2,1,4,3,2,7,4,3} \ , \ \Sigma(68) = \sigma_{3,2,1,4,3,5,4,6,5} \ , \ \Sigma(69) = \sigma_{3,2,1,4,5,6,7,4,5} \\
&\Sigma(70) = \sigma_{3,4,5,6,7,4,3,5,4} \\
&\Sigma(71) = \sigma_{3,2,1,4,3,2,5,4,3,6,5} \\
&\Sigma(72) = \sigma_{3,2,1,4,3,2,5,4,3,6,5,4,7}
\end{aligned}$$

V. REFERENCES

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